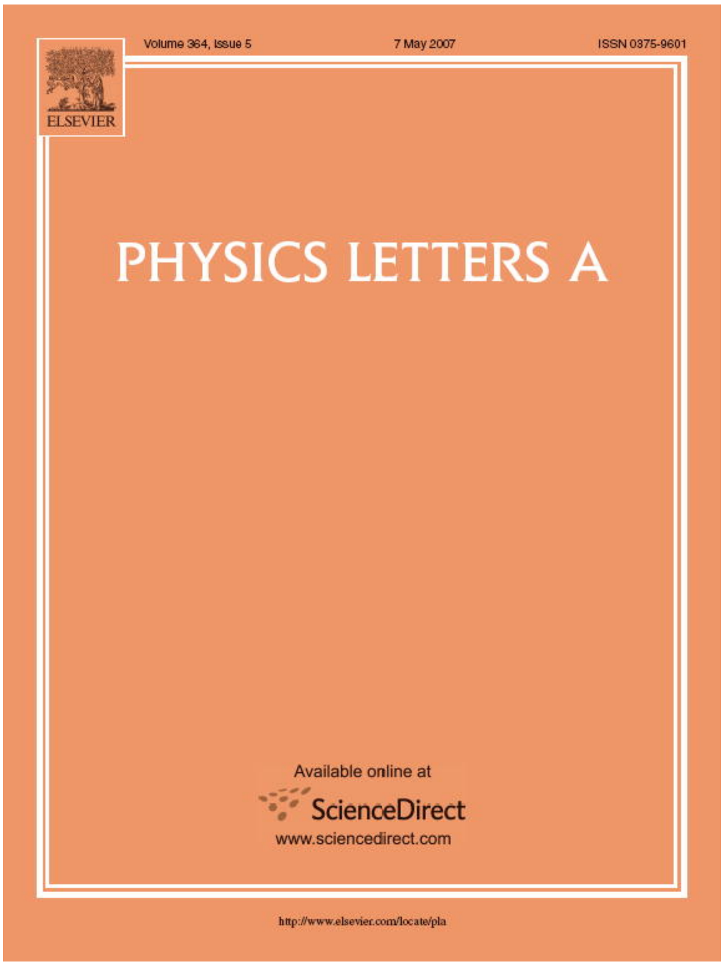
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Nonlinear finite-time Lyapunov exponent and predictability

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Abstract

In this Letter, we introduce a definition of the nonlinear finite-time Lyapunov exponent (FTLE), which is a nonlinear generalization to the existing local or finite-time Lyapunov exponents. With the nonlinear FTLE and its derivatives, the limit of dynamic predictability in large classes of chaotic systems can be efficiently and quantitatively determined.

Keywords: Lyapunov exponent; Nonlinear; Chaos; Logistic map; Predictability

1. Introduction

The prediction of chaotic systems is a significant real-world but challenging open problem [1–3]. By definition, chaotic systems display sensitive dependence on initial conditions: two initially close trajectories can diverge exponentially in the phase space with a rate given by the largest Lyapunov exponent λ_1 [4]. If the initial perturbation is of size δ, and the accepted error tolerance, Δ, is still small, then the largest Lyapunov exponent λ_1 gives a rough estimate of the predictability time: $T_p \sim \frac{1}{\lambda_1}\ln(\frac{\Delta}{\delta})$ [4–7].

However, reliance on the largest Lyapunov exponent, in most situations, proves to be a considerable oversimplification [8]. By and large this is so because the largest Lyapunov exponent, which we will call the global Lyapunov exponent, is defined as the long-term average growth rate of a very small initial error. Often we are not primarily concerned with averages, and, even when we are, we may be interested in short-term behavior. Therefore various local or finite-time Lyapunov exponents are proposed, which measure the short-term growth rate of initial small perturbations [9–13]. Compared with the global Lyapunov exponent, local or finite-time Lyapunov exponents characterize the nonuniform spatial organization and provide information on the variation of predictability on chaotic attractors. But the existing local or finite-time Lyapunov exponents are established on the basis of the infinitesimal uncertainties, which essentially belong to linear error dynamics. Clearly, if an uncertainty is large enough to invalidate the linear error dynamics, it is not applicable anymore to study the predictability of chaotic systems by use of the existing local or finite-time Lyapunov exponents. To determine the limit of predictability, any proposed 'local or finite-time Lyapunov exponent' should be defined with the respect to the nonlinear behaviors of nonlinear dynamical systems [14,15]. Recently, other definitions based on the full nonlinear equations, such as the direct Lyapunov exponent (DLE) [16] and the finite size Lyapunov exponent (FSLE) [17], have been introduced and applied to the analysis of geophysical flows [18–21], giving interesting results.

In this Letter we first give a brief review of linear error dynamics and point out their limitations. Then, we introduce a definition of the nonlinear finite-time Lyapunov exponent (FTLE). The nonlinear FTLE measures the short-term growth rate of initial errors of nonlinear dynamical models without linearizing the governing equations. Finally, we demonstrate the accuracy, validation, and usefulness of the nonlinear FTLE in reflecting the nonlinear behaviors of chaotic systems and quantifying their predictability, by applying it to the logistic map.

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2. Linear error dynamics and limitations

Let us to consider an n-dimensional continuous-time dynamical system

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{F}\big(\mathbf{X}(t)\big), \tag{1}$$

where $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{F}$ is an n-dimensional vector field. Let $\boldsymbol{\delta}(t) = \mathbf{X}(t) - \mathbf{X}_0(t)$ denote deviations from the fiducial orbit $\mathbf{X}_0(t)$. Their evolution equations are given by

$$\frac{d\boldsymbol{\delta}}{dt} = \mathbf{J}(\mathbf{X})\boldsymbol{\delta} + \mathbf{G}(\mathbf{X}, \boldsymbol{\delta}), \tag{2}$$

where $\mathbf{J}(\mathbf{X})\boldsymbol{\delta}$ are the tangent linear terms, $\mathbf{J}(\mathbf{X})$ denotes the $n \times n$ Jacobian matrix, and $\mathbf{G}(\mathbf{X}, \boldsymbol{\delta})$ are the high order nonlinear terms of the perturbations $\boldsymbol{\delta}$. Due to some difficulties in solving the nonlinear problem, previous studies assume that the initial perturbations are sufficiently small such that their evolution can be governed approximately by the linear equations [4]:

$$\frac{d\boldsymbol{\delta}}{dt} = \mathbf{J}(\mathbf{X})\boldsymbol{\delta}. \tag{3}$$

Integrating the linearized equations along the fiducial orbit yields the linear propagator, $\boldsymbol{\mu}(\mathbf{X}_0, t)$, which evolves any infinitesimal initial error $\boldsymbol{\delta}(0)$ forward for a time t to $\boldsymbol{\delta}_\mu(t)$ [11]:

$$\boldsymbol{\delta}_\mu(t) = \boldsymbol{\mu}(\mathbf{X}_0, t)\boldsymbol{\delta}(0), \tag{4}$$

where $\mathbf{X}_0 = \mathbf{X}_0(0)$. Then the largest Lyapunov exponent is defined as:

$$\lambda_1 = \lim_{t\to\infty} \tilde{\lambda}(\mathbf{X}_0, t) = \lim_{t\to\infty} \lim_{\|\delta(0)\|\to 0} \frac{1}{t} \ln \frac{\|\boldsymbol{\delta}_\mu(t)\|}{\|\boldsymbol{\delta}(0)\|}, \tag{5}$$

where $\tilde{\lambda}(\mathbf{X}_0, t)$ is the local or finite-time Lyapunov exponent, which depends on the initial displacement in phase space $\mathbf{X}_0$ and time t. The global Lyapunov exponent λ_1 does not depend on $\mathbf{X}_0$ because of ergodicity [22].

To illustrate the limitations of linear error dynamics, we turn to the case of the Lorenz system [23]. We choose its standard parameter values $\sigma = 10$, $r = 28$, and $b = 8/3$, for which the well-known "butterfly" attractor exists. Fig. 1 shows the linear (with a linear approximation to Eq. (2)) and nonlinear (without any approximation to Eq. (2)) average growth of errors in the Lorenz system. It is illustrated that for the short time intervals, there are only trivial differences between the linear and nonlinear error evolutions. With the increasing time, nonlinear error evolution begins to depart from the linear counterpart and finally reaches saturation. However, linear error evolution is characterized by continuous exponential growth. The results show that the linear error dynamics hold only in the case of initial errors infinitesimal and a finite time interval from initial time, which are not applicable to the description of the process from initially exponential growth to finally reach saturation for sufficiently small errors. Also not applicable to the situations that initial errors are not very small.

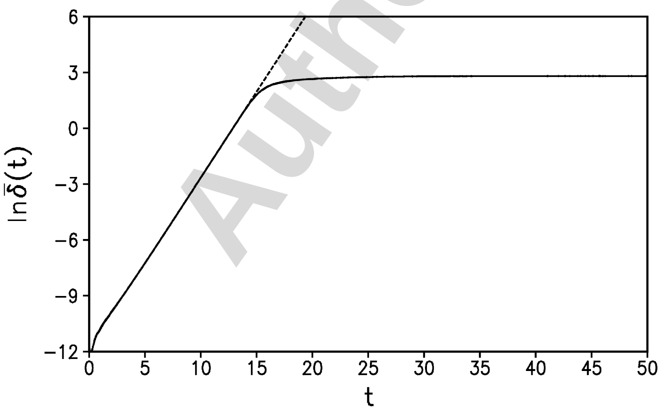

Fig. 1. Linear (dashed line) and nonlinear (solid line) average growth of errors in the Lorenz system as a function of time. The initial magnitude of errors is 10^{-5}.

3. Nonlinear finite-time Lyapunov exponent

Without any approximation, the solutions of Eq. (2) can be obtained by numerically integrating it along the fiducial orbit between times t_0 and t, then

$$\boldsymbol{\delta}_\eta(t) = \boldsymbol{\eta}\big(\mathbf{X}_0, \boldsymbol{\delta}(0), t\big)\boldsymbol{\delta}(0), \tag{6}$$

where $\boldsymbol{\eta}(\mathbf{X}_0, \boldsymbol{\delta}(0), t)$ is defined as the nonlinear propagator, which, as described by Eq. (6), propagates the initial error to the time t in the future. Then the nonlinear finite-time Lyapunov exponent (FTLE) is defined as

$$\lambda\big(\mathbf{X}_0, \boldsymbol{\delta}(0), t\big) = \frac{1}{t} \ln \frac{\|\boldsymbol{\delta}_\eta(t)\|}{\|\boldsymbol{\delta}(0)\|}, \tag{7}$$

where $\lambda(\mathbf{X}_0, \boldsymbol{\delta}(0), t)$ depends in general on the initial displacement in phase space $\mathbf{X}_0$, the initial error $\boldsymbol{\delta}(0)$, and time t, different from the global Lyapunov exponent or the local Lyapunov exponent defined by Eq. (5). For notational simplicity, let the norm of error in phase space at time t be $\delta(t) = \|\boldsymbol{\delta}(t)\|$. The nonlinear FTLE characterizes the growth of an initial perturbation $\boldsymbol{\delta}(0)$ made at point $\mathbf{X}_0$ after a finite time t evolution by the dynamics. For a part of the phase space, the nonlinear FTLE is sometimes positive and sometimes negative (even when the global Lyapunov exponent might be positive). Because of the lack of space, the main emphasis of our research is on the nonlinear characters of the nonlinear FTLE, while a more detailed discussion of local properties of the nonlinear FTLE will be given in the future. We therefore consider here only the average of the exponent over the attractor. As the approximation of this average we use averaging over a great number of orbits started from different initial points on the attractor. To be sure that initial points belong to the attractor we choose them as different points on the same orbit obtained by long time integration of model equations. The mean nonlinear FTLE is given by

$$\bar{\lambda}\big(\boldsymbol{\delta}(0), t\big) = \big\langle \lambda(\mathbf{X}_0, \boldsymbol{\delta}(0), t) \big\rangle_N, \tag{8}$$

where $\langle\ \rangle_N$ denotes the ensemble average of samples of large enough size N ($N \to \infty$). The mean relative growth of initial error (RGIE) can be obtained by

$$\bar{E}(\boldsymbol{\delta}(0), t) = \exp\big(\bar{\lambda}(\boldsymbol{\delta}(0), t)t\big). \tag{9}$$

Main theorem. *Assume that the independent random variables* $X_1, X_2, \ldots, X_n$ *have the following probability distribution*:

$$f(x) = \begin{cases} p(x), & \varepsilon \leqslant x \leqslant a, \\ 0, & x < \varepsilon \text{ or } x > a, \end{cases} \tag{10}$$

where ε *is an arbitrary small positive number, a is a positive constant, and* $p(x)$ *is a continuous function defined on a closed interval* $[\varepsilon, a]$. *Let* $Z_n = (\prod_{i=1}^{n} X_i)^{1/n}$, *then*

$$Z_n \xrightarrow{P} c \quad (n \to \infty), \tag{11}$$

where $\xrightarrow{P}$ *denotes the convergence in probability and c is a constant depending on* $p(x)$.

Proof. A concise proof of the theorem can be given as follows. Firstly we have

$$\ln Z_n = \frac{1}{n} \sum_{i=1}^{n} \ln X_i.$$

Since X_i $(i = 1, 2, \ldots, n)$ follow an independent identically distribution, also do $\ln X_i$ $(i = 1, 2, \ldots, n)$. The mathematical expectation follows that

$$E(\ln X_i) = \int_{\varepsilon}^{a} \ln x \cdot p(x)\, dx = b,$$

where b is a constant depends on $p(x)$. Using the Khinchine's weak law of large numbers [24] as $n \to \infty$, we obtain

$$\ln Z_n \xrightarrow{P} b.$$

Then we have $Z_n \xrightarrow{P} e^b = c$. The proof of our main theorem is completed. □

Remark. From Eqs. (7)–(9), we get

$$\bar{E}\big(\boldsymbol{\delta}(0), t\big) = \exp\left(\frac{1}{N} \sum_{i=1}^{N} \ln \frac{\delta_i(t)}{\delta_i(0)}\right).$$

For the same initial error $\delta(0)$, we have

$$\bar{E}(\boldsymbol{\delta}(0), t) = \left(\prod_{i=1}^{N} \delta_i(t)\right)^{1/N} \Bigg/ \delta(0). \tag{12}$$

For chaotic systems, as $t \to \infty$, $\delta_1(t), \delta_2(t), \ldots, \delta_N(t)$ will follow an independent identically distribution similar to Eq. (10). Consequently, all conditions of the main theorem are satisfied, then we obtain $\bar{E}(\boldsymbol{\delta}(0), t) \xrightarrow{P} c (N \to \infty)$ where c can be considered as the theoretical saturation level of $\bar{E}(\boldsymbol{\delta}(0), t)$. This is called the saturation property of RGIE for chaotic systems. Using the theoretical saturation level, the limit of dynamic predictability can be determined. In addition, for $\bar{\lambda}(\boldsymbol{\delta}(0), t) = \frac{1}{t} \ln[\bar{E}(\boldsymbol{\delta}(0), t)]$, based on the above analysis, we have

$$\bar{\lambda}(\boldsymbol{\delta}(0), t) \xrightarrow{P} \frac{1}{t} \times \ln c \quad \text{as } t \to \infty, \tag{13}$$

so $\bar{\lambda}(\boldsymbol{\delta}(0), t)$ asymptotically decreases like $O(1/t)$ as $t \to \infty$.

4. An example from the logistic map

A simple example is given by the logistic map,

$$y_{n+1} = a y_n (1 - y_n), \quad 0 \leqslant a \leqslant 4. \tag{14}$$

Here we choose the parameter value of $a = 4.0$, for which the logistic map is chaotic on the set (0, 1) [25,26].

Fig. 2 shows the mean nonlinear FTLE and the logarithm of the mean RGIE with $\delta(0) = 10^{-6}$ as a function of time n. As can be seen the mean nonlinear FTLE initially remains a constant and then decreases rapidly after a while and approaches zero as n increases (Fig. 2(a)). It shows that for very small initial error, the growth of error is initially exponential with a growth rate consistent with the largest Lyapunov exponent, implying that linear error dynamics are applicable during this phase. Afterwards the growth of error enters the nonlinear phase with a steadily decreasing growth rate, and finally reaches a saturation value (Fig. 2(b)). At that moment almost all predictability is lost. Following the work of Dalcher and Kalnay [27], we determine the limit of dynamic predictability, which is defined as the time at which error reaches 98% of its saturation level. Then we find that the limit of dynamic predictability of the logistic map with the initial error $\delta(0) = 10^{-6}$ is $n = 18$. In addition, the theoretical saturation level of $\bar{E}(\boldsymbol{\delta}(0), t)$ is found to be completely in line with the actual one (Fig. 2(b)).

Fig. 3 illustrates the dependence of the mean nonlinear FTLE and the mean RGIE on the magnitude of the error. It

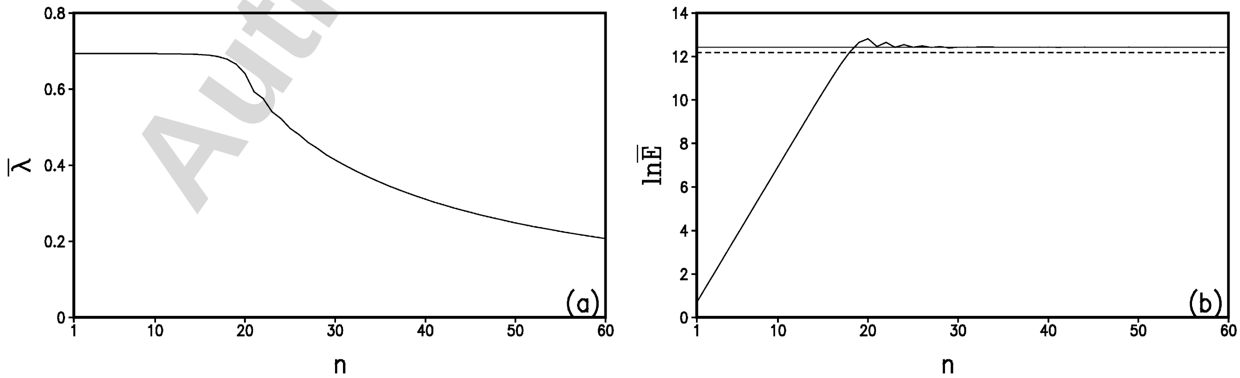

Fig. 2. The mean nonlinear FTLE $\bar{\lambda}(\boldsymbol{\delta}(0), n)$ (a) and the logarithm of the corresponding mean RGIE $\bar{E}(\boldsymbol{\delta}(0), n)$ (b) with $\delta(0) = 10^{-6}$ as a function of time step n. The theoretical saturation level derived from Eq. (12) and the 98% of that are indicated by the constant solid and dashed lines in (b), respectively.

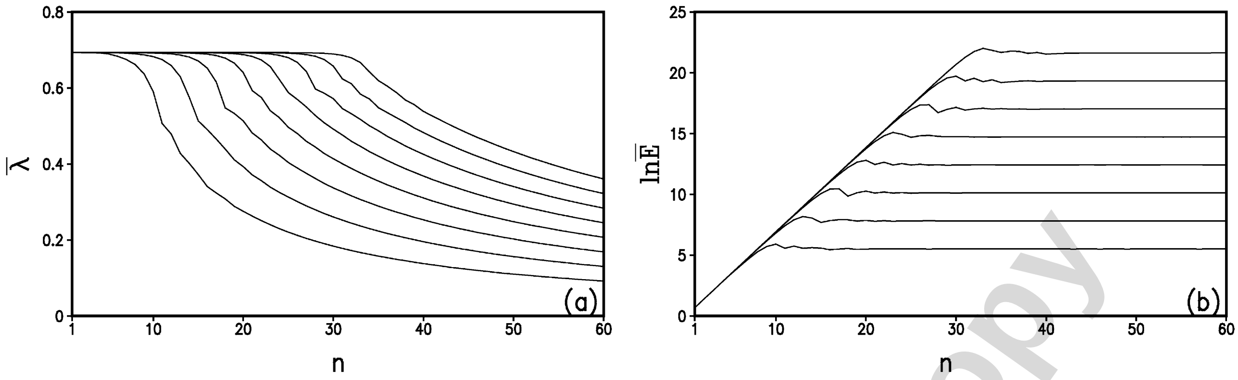

Fig. 3. The mean nonlinear FTLE $\bar{\lambda}(\boldsymbol{\delta}(0), n)$ (a) and the logarithm of the corresponding mean RGIE $\bar{E}(\boldsymbol{\delta}(0), n)$ (b) as a function of time step n and $\delta(0)$ of different magnitude. From above to below, $\delta(0) = 10^{-10}, 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}$, and 10^{-3}, respectively.

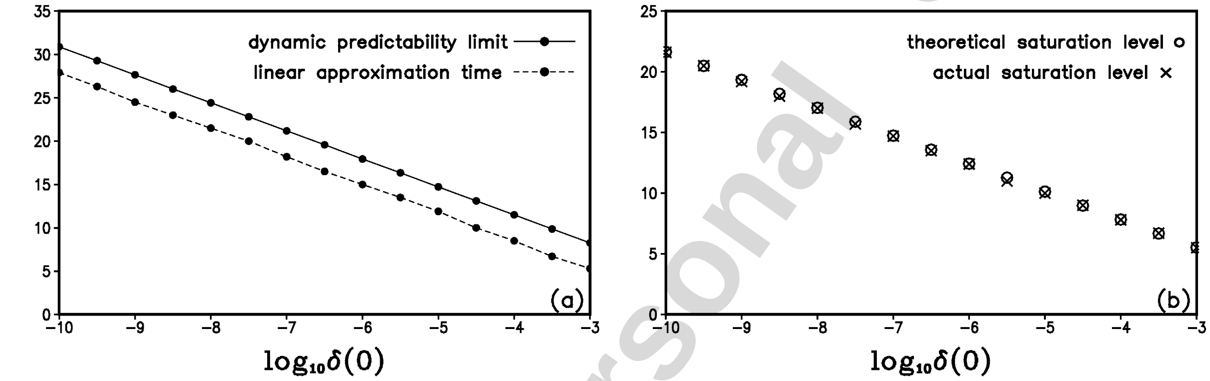

Fig. 4. The limit of dynamic predictability and the time that the tangent linear approximation holds (a), and the theoretical saturation level and the actual saturation level (b) as a function of $\delta(0)$ of different magnitude.

is evident that the mean nonlinear FTLE is essentially constant in a plateau region that widens as the initial error $\delta(0)$ is reduced. For sufficiently large time, however, all the curves show an asymptotic to zero (Fig. 3(a)). Also lengthens the time that the corresponding growth of error reaches saturation as $\delta(0)$ is reduced. However small the initial error $\delta(0)$ be, all the errors finally reach saturation (Fig. 3(b)). The limit of dynamic predictability decreases approximately linearly as $\delta(0)$ is increased (Fig. 4(a)). For a specific initial error, the limit of dynamic predictability is longer than the time that the tangent linear approximation holds, which is defined as the time that the mean nonlinear FTLE remains a constant. The results demonstrate superiority of the nonlinear FTLE in determining the limit of predictability of chaotic systems in comparison with linear one. The theoretical saturation levels are all found good agreement with the actual saturation levels for different initial errors (Fig. 4(b)).

5. Conclusion

We have introduced the definition of nonlinear finite-time Lyapunov exponent (FTLE) and the saturation property of RGIE for chaotic systems, which can be used to efficiently and quantitatively determine the limit of predictability of chaotic systems. The above results are examined by using the simple logistic map. But it is possible that the nonlinear FTLE may be used in a multi-discipline range related to nonlinear dynamics and practical time series analysis, such as the predictability analysis of weather and climate, biological populations, stock market returns, and so on, which will be further subjects of future research.

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